Journal of Approximation Theory **118**, 188–201 (2002)
doi:10.1006/jath.2002.3716

On Some Extremal Properties of Lagrange Interpolatory Polynomials[1]

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Communicated by András Kroó

Received June 25, 2001; accepted in revised form June 26, 2002

In this paper, we will show that Lagrange interpolatory polynomials are optimal for solving some approximation theory problems concerning the finding of linear widths.

In particular, we will show that

$$\inf_{L_n \in \mathscr{L}_n} \sup_{p \in \mathscr{P}_{n+1}} ||p - L_n p||_{\mathbb{C}[-1,1]} = \frac{1}{2^n},$$

where $\mathscr{L}_n$ is a set of the linear operators with finite rank $n+1$ defined on $\mathbb{C}[-1,1]$, and where $\mathscr{P}_{n+1}$ denotes the set of polynomials $p = \sum_{i=0}^{n+1} a_i x^i$ of degree$\leqslant n+1$ such that $|a_{n+1}| \leqslant 1$. The infimum is achieved for Lagrange interpolatory polynomial for nodes $\cos \frac{2k+1}{2(n+1)}$, $k = 0, \ldots, n$.

Key Words: degree of approximation; linear positive operators; operators of class S_m; linear width; finite-dimensional property; Lagrange interpolatory polynomial.

1. INTRODUCTION

Dealing with the problem of approximation of smooth functions by some class of linear operators, we may find that operators of this class have some property which limits the degree of approximation of smooth functions by operators of this class. Let us cite the well-known instances.

It was shown by Korovkin [3] that if linear polynomial operator has the property of positiveness, the degree of approximation of continuous functions by this operator is low. Namely, the degree of approximation by positive linear polynomial operators $L_n f(x)$ ($n \in \mathbb{N}$, $f \in \mathbb{C}[0,1]$, $L_n f$ is

[1]This work is supported by the Russian Foundation for Basic Research, Grants 99-01-01120, 00-15-96123, and by the programme of Ministry of Education of the Russian Federation "Universities of Russia", 990189.

0021-9045/02 $35.00

an algebraical polynomial of degree n) cannot be higher than n^{-2} in $\mathbb{C}[0,1]$ even for the functions $1,\ x,\ x^2$.

Further, in 1962 Korovkin [4] introduced the definition of the operators of class S_m (m is a fixed natural number or zero) and identified [5] the negative property of the operators of class S_m such that the degree of approximation by the linear polynomial operators of class S_m (of degree n) cannot be higher than n^{-m-2} in the norm of uniform convergence even for the functions $1, x, x^2, \dots, x^{m+2}$. Using the idea of Videnskii [8] and Vasiliev has shown in [7] the result of [5] does not depend on the properties of polynomial but rather on the limitation of dimension.

We will need the following definitions and notations.

Let $\mathbb{B}[0,1]$ be the space of real bounded functions with the uniform norm $||f|| = \sup_{x\in[0,1]} |f(x)|$.

Linear operator L_n mapping $\mathbb{C}[0,1]$ into the linear space of finite dimension $n+1$ is called an operator with the finite rank $n+1$.

Following Korovkin [4] say that linear operator L mapping $\mathbb{C}[0,1]$ into $\mathbb{B}[0,1]$ belongs to class S_m (m is a fixed natural number or zero), if for any $x \in [0,1]$ there is function $\psi_x(t) \in \mathbb{C}[0,1]$ that has m or less changes of sign on $[0,1]$ and has the following property: $Lf \geqslant 0$ for any $f \in \mathbb{C}[0,1]$, such that $\operatorname{sgn} f = \operatorname{sgn} \psi_x$.

If $\alpha_{k,n},\ k = 0,1,\dots,n$, are points in $[0,1]$ and $l_{k,n}(x) \in \mathbb{B}[0,1],\ k = 0,1,\dots,n$, then operator

$$L_n f(x) = \sum_{k=0}^{n} f(\alpha_{k,n}) l_{k,n}(x)$$

is a linear operator from $\mathbb{C}[0,1]$ into $\mathbb{B}[0,1]$ which we call $\mathscr{I}$-operator with grid $\alpha = (\alpha_{k,n})_{k=0}^{n}$ and write $L_n \in \mathscr{I}(\alpha)$. This means that the values of the function in a certain finite set of points determine the value of the operator on that function (cf. [1, p. 26]).

Let $\mathscr{I}_{n,m}(\alpha)$ be the set of $\mathscr{I}$-operators of class S_m with grid $\alpha = (0 \leqslant \alpha_{0,n} < \alpha_{1,n} < \cdots < \alpha_{n,n} \leqslant 1)$ mapping $\mathbb{C}[0,1]$ into $\mathbb{B}[0,1]$.

Denote $\mathscr{I}_{n,m} := \bigcup_\alpha \mathscr{I}_{n,m}(\alpha)$. Note, that for $\mathscr{I}$-operators, the condition $L \in S_m$ is tantamount to the following: for any $x \in [0,1]$ the number of sign changes in sequence $l_{k,n}(x),\ k = 0,1,\dots,n$, does not exceed m.

In this paper, we get the exact value of width:

$$\inf_{L_n \in \mathscr{I}_{n,m}} \sup_{p \in \mathscr{P}_{m+2}} ||p - L_n p||, \tag{1.1}$$

where $\mathscr{P}_{m+2}$ denotes the set of polynomials $p = \sum_{i=0}^{m+2} a_i x^i$ of degree $\leqslant m+2$, such that $|a_{m+2}| \leqslant 1$. Note that the linear width (1.1) has "good" class of functions (functions are infinitely smooth) while the class of operators is "bad" (operators are finite dimensional and the number of kernel oscillations is limited to a fixed value).

As a consequence, we will show that the finite-dimensional property is negative in a sense that an error of approximation of such operators does not decrease with the increase of smoothness of approximated functions. Namely, we will show that

$$\inf_{L_n\in\mathscr{L}_n}\ \sup_{p\in\mathscr{P}_{n+1}} ||p-L_np||_{\mathbb{C}[-1,1]}=\frac{1}{2^n},$$

where $\mathscr{L}_n$ is a set of the linear operators with finite rank $n+1$ defined on $\mathbb{C}[-1,1]$, and where $\mathscr{P}_{n+1}$ denotes the set of polynomials $p=\sum_{i=0}^{n+1} a_ix^i$ of degree $\leqslant n+1$ such that $|a_{n+1}|\leqslant 1$. The infimum is achieved for Lagrange interpolatory polynomial for nodes $\cos\frac{2k+1}{2(n+1)},\ k=0,\dots,n$.

2. LEMMAS

We will begin with proof of some lemmas of algebraic nature. The principal lemmas of this section are Lemmas 2.3 and 2.4.

Let $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Let $s^-(x)$ denote a number of sign changes in sequence $x=(x_i)_{i=0}^b\in\mathbb{R}^{b+1}$ (zeros are not taken into account).

LEMMA 2.1. *Let* $b,p\in\mathbb{N};\ a=(a_i)_{i=0}^b\in\mathbb{R}^{b+1},\ a_i\neq 0,\ i=0,\dots,b,$ $a_{i-1}<a_i,\ i=1,\dots,b;\ h_j\in\mathbb{N},\ j=1,\dots,p,\ 0<h_1<\dots<h_p\leqslant b+p.$ *Let* $x=(x_i)_{i=0}^b$ *be the solution of system*

$$\sum_{i=0}^b a_i^l x_i=\begin{cases}1, & l=0,\\ 0, & l=1,\dots,b+p,\ \ l\neq h_j,\ \ j=1,\dots,p.\end{cases}\tag{2.1}$$

Then

(a) $s^-(x)=b-1$, *if there is integer* $1\leqslant r\leqslant b$, *such that* $a_{r-1}<0<a_r$;

(b) $s^-(x)=b$, *if* $a_i>0,\ i=0,\dots,b$ (*or* $a_i<0,\ i=0,\dots,b$).

Proof. The solution of system (2.1) is $x=(x_k)_{k=0}^b$,

$$x_k=(-1)^k\frac{\det||a_i^l||_{i=0,\dots,b,\ i\neq k}^{l=0,\dots,b+p-1,\ l\neq h_j-1,\ j=1,\dots,p}}{\det||a_i^l||_{i=0,\dots,b}^{l=0,\dots,b+p,\ l\neq h_j,\ j=1,\dots,p}}\prod_{i=0,i\neq k}^b a_i.$$

The number of sign changes in sequence $x=(x_k)_{k=0}^b$ is equal to that of sequence

$$\mathrm{sgn}\left\{(-1)^k\prod_{i=0,i\neq k}^b a_i\right\},\qquad k=0,\dots,b.\quad\blacksquare$$

LEMMA 2.2. *Let* $b, p \in \mathbb{N}$; $a = (a_i)_{i=0}^{b} \in \mathbb{R}^{b+1}$, $a_i \neq 0$, $i = 0, \ldots, b$, $a_{i-1} < a_i$, $i = 1, \ldots, b$; $h_j \in \mathbb{N}$, $j = 1, \ldots, p$, $0 < h_1 < \cdots < h_p \leqslant b + p$. *Then for fixed* $j \in \{1, \ldots, p\}$,

$$\sum_{k=0}^{b} (-1)^k a_k^{h_j - 1} \frac{\det||a_i^l||_{i=0,\ldots,b,\ i \neq k}^{l=0,\ldots,b+p-1,\ l \neq h_r - 1,\ r=1,\ldots,p}}{\det||a_i^l||_{i=0,\ldots,b}^{l=0,\ldots,b+p-1,\ l \neq h_r - 1,\ r=1,\ldots,p,\ r \neq j}} = (-1)^{h_j - 1}.$$

Proof. Consider the system

$$\begin{cases} \sum_{k=0}^{b} a_k^l x_k = 0, \quad l = 0, \ldots, b + p - 1,\ l \neq h_r - 1,\ r = 1, \ldots, p, \\ \sum_{k=0}^{b} a_k^{h_j - 1} x_k = (-1)^{h_j - 1}. \end{cases} \tag{2.2}$$

The solution of this system is $x = (x_k)_{k=0}^{b}$,

$$x_k = (-1)^k \frac{\det||a_i^l||_{i=0,\ldots,b,\ i \neq k}^{l=0,\ldots,b+p-1,\ l \neq h_r - 1,\ r=1,\ldots,p}}{\det||a_i^l||_{i=0,\ldots,b}^{l=0,\ldots,b+p-1,\ l \neq h_r - 1,\ r=1,\ldots,p,\ r \neq j}}, \qquad k = 0, \ldots, b. \tag{2.3}$$

The statement of Lemma 2.2 follows immediately from (2.3) and the last equation of (2.2). ∎

Denote $D_m := \{l \in \mathbb{R}^{n+1} :\ s^-(l) \leqslant m\}$.

LEMMA 2.3. *Let* $n > m$, $n \in \mathbb{N}$, $m \in \mathbb{N}_0$, $x \in [0, 1]$. *Then*

$$\inf_{l \in D_m} \left\{ \left| \sum_{k=0}^{n} l_k - 1 \right| + \sum_{r=1}^{m+2} \left| \sum_{k=0}^{n} \left(\frac{k}{n} - x \right)^r l_k \right| \right\} = \min_{0 \leqslant s_0 < \cdots < s_{m+1} \leqslant n,\ s_i \in \mathbb{N}_0} \prod_{i=0}^{m+1} \left| \frac{s_i}{n} - x \right|. \tag{2.4}$$

Proof. Let

$$Z(x, l) := \left| \sum_{k=0}^{n} l_k - 1 \right| + \sum_{r=1}^{m+2} \left| \sum_{k=0}^{n} \left(\frac{k}{n} - x \right)^r l_k \right|, \qquad l = (l_k)_{k=0}^{n} \in \mathbb{R}^{n+1},$$

$$\Delta_m := \{\delta = (\delta_k)_{k=0}^{n} \in \mathbb{R}^{n+1} :\ \delta_k \in \{0, -1, +1\},\ k = 0, \ldots, n,\ s^-(\delta) \leqslant m\},$$

$$D_m(\delta) := \{l = (l_k)_{k=0}^{n} \in \mathbb{R}^{n+1} :\ \operatorname{sgn} l_k = \operatorname{sgn} \delta_k,\ k = 0, \ldots, n\}$$

with $\delta \in \Delta_m$,

$$E := \{\varepsilon = (\varepsilon_r)_{r=0}^{m+2} \in \mathbb{R}^{m+2} :\ \varepsilon_r \in \{-1, +1\},\ r = 0, \ldots, m+2\},$$

$$Z_m(x) := \inf_{l \in D_m} Z(x, l).$$

We have $D_m = \bigcup_{\delta \in \Delta_m} D_m(\delta)$.

For each pair $\delta \in \Delta_m,\ \varepsilon \in E$ let us set the following problem of linear programming:

$$f^{\varepsilon,\delta} = \varepsilon_0 \left(\sum_{k=0}^{n} \delta_k x_k - 1 \right) + \sum_{r=1}^{m+2} \varepsilon_r \sum_{k=0}^{n} \left(\frac{k}{n} - x \right)^r \delta_k x_k \to \min,$$

$$\begin{cases} \varepsilon_0 \left(\sum_{k=0}^{n} \delta_k x_k - 1 \right) - y_0 = 0, \\ \varepsilon_r \sum_{k=0}^{n} \left(\frac{k}{n} - x \right)^r \delta_k x_k - y_r = 0, \quad r = 1, \ldots, m+2, \\ x_k \geqslant 0, \quad k = 0, \ldots, n, \\ y_r \geqslant 0, \quad r = 0, \ldots, m+2. \end{cases} \tag{2.5}$$

Note that a number of such pair $\delta \in \Delta_m,\ \varepsilon \in E$ is finite.

We will use terminology and facts of the theory of linear and convex programming.

A feasible set is a set of points $(x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_{m+2}) \in \mathbb{R}^{n+m+4}$, which satisfy the system of constraints (2.5). Let $f^{\varepsilon,\delta}_{\min}$ denote a solution of problem $f^{\varepsilon,\delta} \to \min$. If for some ε, δ the feasible set of problem $f^{\varepsilon,\delta} \to \min$ is empty we put $f^{\varepsilon,\delta}_{\min} = +\infty$.

We have

$$Z_m(x) := \inf_{l \in D_m} Z(x,l) = \min_{\delta \in \Delta_m} \inf_{l \in D_m(\delta)} Z(x,l) = \min_{\delta \in \Delta_m} \min_{\varepsilon \in E} f^{\varepsilon,\delta}_{\min}. \tag{2.6}$$

The rest of the lemma is devoted to finding the right-hand side of Eq. (2.6).

Let $\varepsilon \in E,\ \delta \in \Delta_m$ be such that a feasible set of problem $f^{\varepsilon,\delta} \to \min$ is not empty (if for any $\varepsilon \in E$ and $\delta \in \Delta_m$ a feasible set of problem $f^{\varepsilon,\delta} \to \min$ is empty, then D_m is empty too that is impossible). It follows from the main theorem of the linear programming [2] that the solution of problem $f^{\varepsilon,\delta} \to \min$ is achieved in one of the extreme points of the feasible set, and this extreme point is determined by the choice of $m+3$ basic variables.

Let the extreme point $(x_0^*, \ldots, x_n^*, y_0^*, \ldots, y_{m+2}^*)$ of the feasible set of problem $f^{\varepsilon,\delta} \to \min$ be determined by the choice of basic variables x_{s_k}, y_{h_j}, $0 \leqslant s_0 < \cdots < s_{m+2-p} \leqslant n$, $0 \leqslant h_1 < \cdots < h_p \leqslant m+2$, $p \in \mathbb{N}$, $1 \leqslant p \leqslant m+2$.

Let us consider cases $h_1 = 0$ and $\neq 0$ separately.

(I) If $h_1 = 0$ then for $l^* = (l_i^*)_{i=0}^n = (\delta_i x_i^*)_{i=0}^n$ we have $l_i^* = 0, i = 0, \ldots, n$, $i \neq s_k$, $k = 0, \ldots, m+2-p$; and values $l_{s_k}^*$, $k = 0, \ldots, m+2-p$, must satisfy system

$$\left\{ \sum_{k=0}^{m+2-p} \left(\frac{s_k}{n} - x\right)^r l^*_{s_k} = 0, \qquad r = 1, \ldots, m+2, \quad r \neq h_j, \ j = 2, \ldots, p. \right.$$

Then $l^*_k = 0, \ k = 0, \ldots, n$, and $f^{\varepsilon,\delta}_{\min} = 1$.

(II) If $h_1 \neq 0$ then for $l^* = (l^*_i)^n_{i=0} = (\delta_i x^*_i)^n_{i=0}$ we have $l^*_i = 0, i = 0, \ldots, n$, $i \neq s_k, \ k = 0, \ldots, m+2-p$; and values $l^*_{s_k}, \ k = 0, \ldots, m+2-p$, must satisfy the system

$$\begin{cases} \sum_{k=0}^{m+2-p} l^*_{s_k} = 1, \\ \sum_{k=0}^{m+2-p} \left(\frac{s_k}{n} - x\right)^r l^*_{s_k} = 0, \quad r = 1, \ldots, m+2, \ r \neq h_j, \ j = 1 \ldots, p. \end{cases}$$

Note that $p \geqslant 1$. Indeed if $p = 0$, then it follows from Lemma 2.1 that $s^-(l^*) \geqslant m+1$. This contradicts $\delta \in \Delta_m$.

We have

$$l^*_{s_k} = (-1)^k \frac{d_k}{d}, \qquad k = 0, \ldots, m+2-p$$

with

$$d_k = \det \left\| \left(\frac{s_i}{n} - x\right)^l \right\|^{l=1,\ldots,m+2, \ l \neq h_j, \ j=1,\ldots p,}_{i=0,\ldots,m+2-p, \ i \neq k,}$$

$$d = \det \left\| \left(\frac{s_i}{n} - x\right)^l \right\|^{l=0,\ldots,m+2, \ l \neq h_j, \ j=1,\ldots,p,}_{i=0,\ldots,m+2-p.}$$

For $h_j, \ j = 1, \ldots, p$, we have

$$\begin{aligned}
&\varepsilon_{h_j} \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^{h_j} \delta_k x_k \\
&= \left| \sum_{k=0}^{m+2-p} \left(\frac{s_k}{n} - x\right)^{h_j} l^*_{s_k} \right| = \left| \sum_{k=0}^{m+2-p} (-1)^k \left(\frac{s_k}{n} - x\right)^{h_j} \frac{d_k}{d} \right| \\
&= \left| \frac{\det \|(\frac{s_i}{n} - x)^l\|^{l=1,\ldots,m+2, \ l \neq h_r, \ r=1,\ldots,p, \ r \neq j}_{i=0,\ldots,m+2-p}}{\det \|(\frac{s_i}{n} - x)^l\|^{l=0,\ldots,m+2, \ l \neq h_r, \ r=1,\ldots,p}_{i=0,\ldots,m+2-p}} \right| \\
&\quad \times \left| \sum_{k=0}^{m+2-p} (-1)^k \left(\frac{s_k}{n} - x\right)^{h_j - 1} \frac{\det \|(\frac{s_i}{n} - x)^l\|^{l=0,\ldots,m+1, \ l \neq h_r - 1, \ r=1,\ldots,p}_{i=0,\ldots,m+2-p, \ i \neq k}}{\det \|(\frac{s_i}{n} - x)^l\|^{l=0,\ldots,m+1, \ l \neq h_r - 1, \ r=1,\ldots,p, \ r \neq j}_{i=0,\ldots,m+2-p}} \right|.
\end{aligned}$$

It follows from Lemma 2.2 that the sum placed inside the modulus equals $(-1)^{h_j-1}$. Since

$$\varepsilon_i \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^i \delta_k x_k = 0, \qquad i = 0, \dots, m+2, \quad i \neq h_j, \; j = 1, \dots, p,$$

we get that the value of linear function of problem $f^{\varepsilon,\delta} \to \min$ in extreme point $(x_0^*, \dots, x_n^*, \; y_0^*, \dots, y_{m+2}^*)$ equals

$$\sum_{r=1}^{m+2} \varepsilon_r \left(\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^r \delta_k x_k^* \right) = \sum_{v=1}^{p} \left| \frac{\det \|(\frac{s_i}{n} - x)^l\|_{i=0,\dots,m+2-p}^{l=1,\dots,m+2, \; l\neq h_j, \; j=1,\dots,p, \; j\neq v}}{\det \|(\frac{s_i}{n} - x)^l\|_{i=0,\dots,m+2-p}^{l=0,\dots,m+2, \; l\neq h_j, \; j=1,\dots,p}} \right|.$$

Now, taking into account (I) and (II) we can conclude that

$$Z_m(x) = \min\left\{ 1, \; \min_{t, j_h, s_k} \frac{\sum_{v=1}^{m+2} |\det \|(\frac{s_i}{n} - x)^l\|_{i=0,\dots,t}^{l=v; \; l=j_h, \; h=1,\dots t}|}{|\det \|(\frac{s_i}{n} - x)^l\|_{i=0,\dots,t}^{l=0; \; l=j_h, \; h=1,\dots,t}|} \right\},$$

where the minimum is found in

(1) $1 \leqslant t \leqslant m+1, \; 0 < j_1 < j_2 < \cdots < j_t < m+2, \; 0 \leqslant s_0 < \cdots < s_t \leqslant n,$

(2) $t = m, \; 0 < j_1 < j_2 < \cdots < j_t < m+2, \; 0 \leqslant s_0 < \cdots < s_t \leqslant n,$ such that $\operatorname{sgn}(\frac{s_i}{n} - x) = \operatorname{sgn}(\frac{s_j}{n} - x)$ for all $i, j = 0, 1, \dots, t$ (see Lemma 2.1(b)).

Finally, using determinant properties we get

$$Z_m(x) = \min_{0 \leqslant s_0 < s_1 < \cdots < s_{m+1} \leqslant n} \prod_{i=0}^{m+1} \left| \frac{s_i}{n} - x \right|. \quad \blacksquare$$

By repeating the proof of Lemma 2.3 for grid $\alpha = (0 \leqslant \alpha_{0,n} < \alpha_{1,n} < \cdots < \alpha_{n,n} \leqslant 1)$ instead of $(\frac{k}{n})_{k=0}^n$ we obtain the following lemma.

LEMMA 2.4. *Let* $n > m, \; n \in \mathbb{N}, \; m \in \mathbb{N}_0, \; x \in [0,1]$. *Then*

$$\inf_{l \in D_m} \left\{ \left| \sum_{k=0}^{n} l_k - 1 \right| + \sum_{r=1}^{m+2} \left| \sum_{k=0}^{n} (\alpha_{k,n} - x)^r l_k \right| \right\} = \min_{0 \leqslant s_0 < \cdots < s_{m+1} \leqslant n, \; s_i \in \mathbb{N}_0} \prod_{i=0}^{m+1} |\alpha_{s_i,n} - x|.$$

Let $\mathscr{L}_n$ be the set of linear operators with finite rank $n+1$ defined on $\mathbb{C}[0,1]$. Let us write $\mathscr{I}_n = \mathscr{I}_{n,n}$.

LEMMA 2.5. *Let* $g \in \mathbb{C}[0,1]$ *and* $L_n \in \mathscr{L}_n$. *Then there is operator* $M_n \in \mathscr{I}_n$, *such that* $L_n g = M_n g$.

Proof. Indeed, it follows from the Riesz representation theorem that every linear operator $L_n \in \mathscr{L}_n$ can be represented by

$$L_n f(x) = \sum_{k=0}^{n} \left\{ \int_0^1 f \, d\mu_{k,n} \right\} u_{k,n}(x), \qquad f \in \mathbb{C}[0,1],$$

where $(d\mu_{k,n})_{k=0}^{n}$ is the system of measures on $[0,1]$, and functions $(u_{k,n})_{k=0}^{n}$ are the generating linear space $\{L_n f : f \in \mathbb{C}[0,1]\}$. Without loss of generality, we assume that $\int_0^1 d\mu_{k,n} = 1, \ k = 0, 1, \ldots, n$.

On the other hand, it follows from the mean value theorem that there is grid $(\alpha_{k,n})_{k=0}^{n} \subset [0,1]$, such that

$$\int_0^1 g \, d\mu_{k,n} = g(\alpha_{k,n}), \qquad k = 0, 1, \ldots, n.$$

Thus, the linear operator

$$M_n f(x) = \sum_{k=0}^{n} f(\alpha_{k,n}) u_{k,n}(x), \qquad f \in \mathbb{C}[0,1],$$

belongs to the set $\mathscr{I}_n$ and $L_n g = M_n g$. ∎

Let $\mathscr{L}_{n,0}$ denotes the set of linear positive operators with finite rank $n+1$ defined on $\mathbb{C}[0,1]$.

LEMMA 2.6. *Let* $g \in \mathbb{C}[0,1]$ *and* $L_n \in \mathscr{L}_{n,0}$. *Then there is operator* $M_n \in \mathscr{I}_{n,0}$, *such that* $L_n g = M_n g$.

The proof of this lemma is similar to the previous one, only that functions $u_{k,n}(x) \geqslant 0, \ k = 0, \ldots, n, \ x \in [0,1]$.

3. THE MAIN RESULTS

The main results of this paper can be stated as follows.

THEOREM 3.1. *Let* $x \in [0,1]$ *and let* $\mathscr{P}_{m+2}$ *be the space of polynomial* $p = \sum_{i=0}^{m+2} a_i x^i$ *of degree* $\leqslant m+2$, *such that* $|a_{m+2}| \leqslant 1$. *Let* $\alpha = (0 \leqslant \alpha_{0,n} < \cdots < \alpha_{n,n} \leqslant 1)$. *Then*

$$\inf_{L_n\in\mathcal{I}_{n,m}(\alpha)} \sup_{p\in\mathcal{P}_{m+2}} |p(x)-L_np(x)| = \min_{0\leqslant s_0<\cdots<s_{m+1}\leqslant n} \prod_{i=0}^{m+1} |\alpha_{s_i,n}-x|. \qquad (3.1)$$

Proof. Since $p\in\mathcal{P}_{m+2}$ we have

$$p(t)=\sum_{r=0}^{m+2}\frac{p^{(r)}(x)}{r!}(t-x)^r, \qquad t,x\in[0,1].$$

Let $\mathcal{P}^0_{m+2}$ denote the subset of $\mathcal{P}_{m+2}$ which consists of polynomials p, such that

$$||p^{(r)}||\leqslant r!, \qquad r=0,1,\ldots,m+2.$$

It is obvious that

$$\inf_{L_n\in\mathcal{I}_{n,m}(\alpha)} \sup_{p\in\mathcal{P}_{m+2}} |p(x)-L_np(x)|\geqslant \inf_{L_n\in\mathcal{I}_{n,m}(\alpha)} \sup_{p\in\mathcal{P}^0_{m+2}} |p(x)-L_np(x)|.$$

For each operator $L_n\in\mathcal{I}_{n,m}(\alpha)$ there are functions $l_{k,n}(x)\in\mathbb{B}[0,1]$, $k=0,\ldots,n$, such that

$$L_nf(x)=\sum_{k=0}^{n} f(\alpha_{k,n})l_{k,n}(x), \qquad f\in\mathbb{C}[0,1],$$

and for each $x\in[0,1]$ a number of sign changes of sequence $(l_{k,n}(x))_{k=0}^n$ is less than or equal to m. Then

$$\begin{aligned}
&\inf_{L_n\in\mathcal{I}_{n,m}(\alpha)} \sup_{p\in\mathcal{P}^0_{m+2}} |p(x)-L_np(x)| \\
&= \inf_{L_n\in\mathcal{I}_{n,m}(\alpha)} \sup_{p\in\mathcal{P}^0_{m+2}} \left|p(x)-L_n\left(\sum_{r=0}^{m+2}\frac{p^{(r)}(x)}{r!}(t-x)^r\right)(x)\right| \\
&= \inf_{L_n\in\mathcal{I}_{n,m}(\alpha)} \left(|L_n(1;x)-1|+\sum_{r=1}^{m+2}|L_n((t-x)^r)(x)|\right) \\
&= \inf_{(l_{k,n}(x))_{k=0}^n\in D_m} \left(\left|\sum_{k=0}^{n} l_{k,n}(x)-1\right|+\sum_{r=1}^{m+2}\left|\sum_{k=0}^{n}(\alpha_{k,n}-x)^r l_{k,n}(x)\right|\right).
\end{aligned}$$

It follows from Lemma 2.4 that

$$\inf_{L_n\in\mathcal{I}_{n,m}(\alpha)} \sup_{p\in\mathcal{P}_{m+2}} |p(x)-L_np(x)|\geqslant \min_{0\leqslant s_0<\cdots<s_{m+1}\leqslant n} \prod_{i=0}^{m+1} |\alpha_{s_i,n}-x|.$$

On the other hand, let us consider the linear operator

$$M_{n,m}f(x) = \sum_{k=0}^{n} f(\alpha_{k,n})\mu_{k,n,m}(x),$$

where functions $\mu_{k,n,m}(x),\ k = 0, \dots, n$, satisfy

$$\sum_{j \in J_k} (\alpha_{j,n} - x)^r \mu_{j,n,m}(x) = \begin{cases} 1, & r = 0, \\ 0, & r = 1, \dots, m+1, \end{cases}$$

$$x \in [\alpha_{k,n}, \alpha_{k+1,n}), \qquad k = 0, \dots, n-1,$$

where

$$J_k := \Bigg\{ 0 \leqslant j_0 < \cdots < j_{m+1} \leqslant n \ : \ \sup_{x \in [\alpha_{k,n}, \alpha_{k+1,n}]} \prod_{i=0}^{m+1} |\alpha_{j_i,n} - x| = \min_{0 \leqslant s_0 < \cdots < s_{m+1} \leqslant n} \sup_{x \in [\alpha_{k,n}, \alpha_{k+1,n}]} \prod_{i=0}^{m+1} |\alpha_{s_i,n} - x| \Bigg\}.$$

It follows from Lemma 2.1 that $M_{n,m} \in S_m$. It follows from Lemma 2.2 that for $x \in [\alpha_{k,n}, \alpha_{k+1,n})$,

$$\sup_{p \in \mathscr{P}_{m+2}} |p(x) - M_{n,m}p(x)| = \min_{0 \leqslant s_0 < \cdots < s_{m+1} \leqslant n} \prod_{i=0}^{m+1} |\alpha_{s_i,n} - x|. \quad \blacksquare$$

Remark. Note that on $[\alpha_{\nu-1,n}, \alpha_{\nu,n}],\ \nu = 1, \dots, n$, polynomial $M_{n,m}f(x)$ coincides with Lagrange interpolatory polynomial for nodes $\alpha_{j,n}$, $j \in J_\nu$.

THEOREM 3.2. *Let* $\alpha^* = (\frac{k}{n})_{k=0}^{n}$. *Then*

$$\inf_{L_n \in \mathscr{I}_{n,m}(\alpha^*)} \sup_{p \in \mathscr{P}_{m+2}} ||p - L_n p|| = \frac{C(m)}{n^{m+2}}, \tag{3.2}$$

where $C(m) = \sup_{x \in [0,1]} \prod_{i=0}^{m+1} |i - x|$.

Proof. It follows from Theorem 3.1 that

$$\begin{aligned}
&\inf_{L_n \in \mathscr{I}_{n,m}(\alpha^*)} \sup_{p \in \mathscr{P}_{m+2}} ||p - L_n p|| \\
&\geqslant \sup_{x \in [0,1]} \inf_{L_n \in \mathscr{I}_{n,m}(\alpha^*)} \sup_{p \in \mathscr{P}_{m+2}} |p(x) - L_n p(x)| \\
&= \sup_{x \in [0,1]} \min_{0 \leqslant s_0 < \cdots < s_{m+1} \leqslant n,\ s_i \in \mathbb{N}_0} \prod_{i=0}^{m+1} \left| \frac{s_i}{n} - x \right| \\
&= \sup_{x \in [0, \frac{1}{n}]} \prod_{i=0}^{m+1} \left| \frac{i}{n} - x \right| = \frac{C(m)}{n^{m+2}}.
\end{aligned}$$

On the other hand, let us consider the linear operator $\Lambda_{n,m} : \mathbb{C}[0,1] \to \mathbb{C}[0,1]$ defined by

$$\begin{aligned}
\Lambda_{n,m} f(x) = {} & \sum_{i=0}^{m+1} f\left(\frac{i + v - [\frac{m+3}{2}] + s - t}{n} \right) (-1)^i \frac{1}{i!(m+1-i)!} \\
& \times \prod_{r=0, r \neq i}^{m+1} \left(r + v - \left[\frac{m+3}{2} \right] - nx + s - t \right),
\end{aligned}$$

$$x \in \left[\frac{v-1}{n}, \frac{v}{n} \right], \qquad v = 1, \ldots, n, \quad m, n \in \mathbb{N}_0, \ \ n > m, \ \ f \in \mathbb{C}[0,1], \tag{3.3}$$

where $s, t \in \mathbb{N}_0$ defined by

$$s = \begin{cases} [\frac{m+3}{2}] - v & \text{if } v = 1, \ldots, [\frac{m+3}{2}] - 1, \\ 0 & \text{for other } v, \end{cases}$$

$$t = \begin{cases} v - n + [\frac{m}{2}] & \text{if } v = n - [\frac{m}{2}] + 1, \ldots, n, \\ 0 & \text{for other } v. \end{cases}$$

For each $x \in [0,1]$ we have

$$\sum_{k=0}^{n} \left(\frac{k}{n} - x \right)^r \lambda_{k,n,m}(x) = \begin{cases} 1, & r = 0 \\ 0, & r = 1, \ldots, m+1. \end{cases}$$

It follows from Lemma 2.1 that $\Lambda_{n,m} \in \mathscr{I}_{n,m}(\alpha)$. We have

$$\begin{aligned}
&\sup_{p\in\mathscr{P}_{m+2}} \|p - \Lambda_{n,m}p\| \\
&\quad = \sup_{p\in\mathscr{P}_{m+2}} \sup_{x\in[0,1]} \left| \sum_{r=1}^{m+2} \frac{p^{(r)}(x)}{r!} \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^r \lambda_{k,n,m}(x) \right| \\
&\quad = \max_{1\leqslant k\leqslant n} \sup_{x\in[\frac{k-1}{n},\frac{k}{n}]} \left| \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^{m+2} \lambda_{k,n,m}(x) \right| \\
&\quad = \sup_{x\in[0,\frac{1}{n}]} \left| \prod_{i=0}^{m+1} \left(\frac{i}{n} - x\right) \right| \left| \sum_{k=0}^{m+1} \frac{(-1)^k}{k!(m+1-k)!} (k - nx)^{m+1} \right|.
\end{aligned}$$

It follows from Lemma 2.2 that

$$\sup_{p\in\mathscr{P}_{m+2}} \| p - \Lambda_{n,m}p\| = \frac{C(m)}{n^{m+2}}.$$

This completes the proof of Theorem 3.2. ∎

Remark. Note that on $[\frac{\nu-1}{n}, \frac{\nu}{n}]$, $\nu = 1, \ldots, n$, polynomial $\Lambda_{n,m}f(x)$ coincides with Lagrange interpolatory polynomial for nodes $\frac{i+\nu-[\frac{m+3}{2}]+s-t}{n}$, $i = 0, \ldots, m+1$, where s, t are defined above.

The following theorem can be proved similarly.

THEOREM 3.3. *Let* $\mathscr{I}_{n,m} := \bigcup_\alpha \mathscr{I}_{n,m}(\alpha)$. *Then*

$$\begin{aligned}
&\inf_{L_n\in\mathscr{I}_{n,m}} \sup_{p\in\mathscr{P}_{m+2}} \|p - L_np\| \\
&\quad = \inf_{0\leqslant\alpha_{0,n}<\cdots<\alpha_{n,n}\leqslant 1} \min_{0\leqslant s_0<\cdots<s_{m+1}\leqslant n} \sup_{x\in[0,1]} \prod_{i=0}^{m+1} |\alpha_{s_i,n} - x|.
\end{aligned}$$

4. COROLLARIES

COROLLARY 4.1. *Let* $\mathscr{L}_n$ *be the set of linear operators* $L_n : \mathbb{C}[-1,1] \to \mathbb{B}[-1,1]$ *with finite rank* $n+1$. *Let* $\mathscr{P}_{n+1}$ *be the space of polynomial* $p = \sum_{i=0}^{n+1} a_i x^i$ *of degree* $\leqslant n+1$, *such that* $|a_{n+1}| \leqslant 1$. *Then*

$$\inf_{L_n\in\mathscr{L}_n} \sup_{p\in\mathscr{P}_{n+1}} \|p - L_np\|_{\mathbb{C}[-1,1]} = \frac{1}{2^n}.$$

Proof. It follows from Lemma 2.5, Theorem 3.3 and [6] that

$$\begin{aligned}
&\inf_{L_n\in\mathscr{L}_n}\ \sup_{p\in\mathscr{P}_{n+1}} \|p-L_np\|_{\mathbb{B}[-1,1]}\\
&\quad=\inf_{L_n\in\mathscr{I}_{n,n}}\ \sup_{p\in\mathscr{P}_{n+1}} \|p-L_np\|_{\mathbb{B}[-1,1]}\\
&\quad=\inf_{-1\leqslant\alpha_{0,n}<\cdots<\alpha_{n,n}\leqslant 1}\ \sup_{x\in[-1,1]} \prod_{i=0}^{n} |\alpha_{i,n}-x|\\
&\quad=\sup_{x\in[-1,1]} \prod_{i=0}^{n} \left|\cos\frac{2i+1}{2n}\pi - x\right| = \frac{1}{2^n}. \quad ∎
\end{aligned}$$

Remark. Note that Lagrange interpolatory polynomial for the nodes $x_k=\cos\frac{2k+1}{2(n+1)}\pi$, $k=0,\ldots,n$,

$$L_n^*f(x)=\sum_{k=0}^{n} f(x_k)\,\frac{(-1)^k T_{n+1}(x)\sqrt{1-x_k^2}}{(n+1)(x-x_n)},$$

where $T_{n+1}(x)=\cos(n+1)\arccos x$, has the following properties:

(a) if $p\in\mathscr{P}_n$, then $L_n^*p(x)\equiv p(x)$ on $[0,1]$;

(b) if $p(x)=x^{n+1}$, then $p(x)-L_n^*p(x)\equiv\frac{1}{2^n}T_{n+1}(x)$ on $[0,1]$. Consequently,

$$\sup_{p\in\mathscr{P}_{n+1}} \|p-L_n^*p\| = \frac{1}{2^n}.$$

COROLLARY 4.2. *Let $\mathscr{L}_{n,0}$ be the set of linear positive operators L_n : $\mathbb{C}[0,1]\to\mathbb{B}[0,1]$ with finite rank $n+1$. Let $\mathscr{P}_2$ be the space of polynomial $p=\sum_{i=0}^{2} a_i\,x^i$ of degree $\leqslant 2$, such that $|a_2|\leqslant 1$. Then*

$$\inf_{L_n\in\mathscr{L}_{n,0}}\ \sup_{p\in\mathscr{P}_2} \|p-L_np\| = \frac{1}{4n^2}.$$

The proposition follows from Lemma 2.6, Theorem 3.3 and extends the results of Korovkin [3] and Videnskii [8].

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